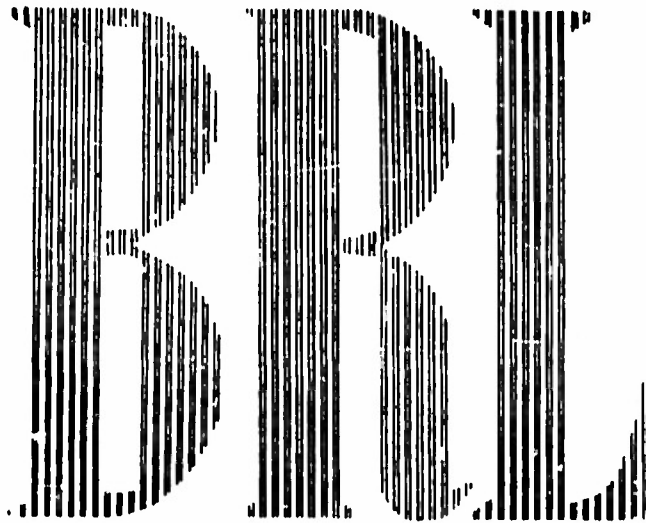


AD No. 18327

ASTIA FILE COPY



REPORT No. 866

COPY NO. \_\_\_\_\_  
Joint Army-Navy Mailing  
List for Guided Missiles  
Dated \_\_\_\_\_

**The Effect of Thickness on Airfoils with  
Constant Vertical Acceleration  
at Supersonic Speeds**

JOHN C. MARTIN  
NATHAN GERBER

BALLISTIC RESEARCH LABORATORIES



ABERDEEN PROVING GROUND, MARYLAND

BALLISTIC RESEARCH LABORATORIES

REPORT NO. 866

May 1953

THE EFFECT OF THICKNESS ON AIRFOILS WITH CONSTANT VERTICAL  
ACCELERATION AT SUPERSONIC SPEEDS

John C. Martin

Nathan Gerber

Project No. TB3-0108H of the Research and  
Development Division, Ordnance Corps

ABERDEEN PROVING GROUND, MARYLAND

# SYMBOLS

$a$	parameter associated with the transition surface attached to the airfoil's leading edge
$a_o$	arbitrary constant
$c$	velocity of sound
$c_r$	airfoil chord
$C_p$	pressure coefficient --- $\left[ \frac{\text{pressure}}{(1/2)\rho v^2} \right]$
$d$	distance from airfoil leading edge to axis of pitch
$f$	arbitrary function associated with the equation of the airfoil surface.
$F$	right hand side of eq. (11)
$g_1, g_2, g_3$	functions associated with the form of the first order potential function
$h_1, h_2, \dots, h_6$	functions associated with the form of the second order potential function
$i, k$	unit vectors
$I_o, I_1, I_2$	functions associated with the form of $F$
$l$	a distance small compared to unity
$M$	Mach number
$q$	rate of pitch
$R$	$\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2(z-\zeta)^2}$
$s$	a function associated with the airfoil surface
$S_o$	surface of integration (see eq. (12))
$t, t'$	time

$t_1 =$	$t - \frac{(x-\xi) M}{\beta^2 c} + \frac{R}{\beta^2 c}$
$t_2 =$	$t - \frac{(x-\xi) M}{\beta^2 c} - \frac{R}{\beta^2 c}$
$\mathbf{v}$	velocity vector
$\mathbf{v}_0$	volume of integration (see eq. (12))
$V$	free stream velocity
$x', y', z'$	rectangular coordinates fixed in space
$x, y, z$	rectangular coordinates attached to airfoil
$\alpha$	angle of attack
$\alpha_m$	maximum angle of attack of an oscillating airfoil
$\ddot{\alpha}$	acceleration
$\beta =$	$\sqrt{M^2 - 1}$
$\gamma$	adiabatic exponent
$\delta$	a small distance ( see fig. 11)
$\epsilon$	thickness parameter
$\xi, \eta, \zeta$	rectangular coordinates
$\theta_1, \theta_2$	auxiliary functions used in finding the second order potential function
$\rho$	density
$\sigma$	parameter associated with the transition surface attached to the airfoil's leading edge
$\phi$	first order potential function
$\phi'$	potential function (see eq. (A-4))
$\Phi$	second order potential function

$\psi$	second order perturbation potential function	
$\omega$	frequency of oscillation	
$C_L$	normal force coefficient	$\left[ \frac{\text{normal force}}{(1/2)\rho V^2 c_r} \right]$
$C_m$	moment coefficient	$\left[ \frac{\text{moment}}{(1/2)\rho V^2 c_r^2} \right]$
$C_{L\dot{\alpha}}$	$= \left[ \frac{\partial C_L}{\partial \dot{\alpha}} \left\{ \dot{\alpha} c_r / (2V) \right\} \right]_{\dot{\alpha} \rightarrow 0}$	
$C_{m\dot{\alpha}}$	$= \left[ \frac{\partial C_m}{\partial \dot{\alpha}} \left\{ \dot{\alpha} c_r / (2V) \right\} \right]_{\dot{\alpha} \rightarrow 0}$	
$C_{mq}$	$= \left[ \frac{\partial C_m}{\partial q} \left\{ q c_r / (2V) \right\} \right]_{q \rightarrow 0}$	

BALLISTIC RESEARCH LABORATORIES

REPORT NO. 866

JCMartin/NGerber/ekb  
Aberdeen Proving Ground, Md.  
May 1953

THE EFFECT OF THICKNESS ON AIRFOILS WITH CONSTANT VERTICAL  
ACCELERATION AT SUPERSONIC SPEEDS

ABSTRACT

The effects of thickness on the lift and pitching-moment of two dimensional airfoils at supersonic speeds with constant vertical acceleration are investigated. The airfoils considered have arbitrary symmetrical cross sections, and the flow is supersonic everywhere.

The analysis is based on a second order theory similar to the second order theory introduced by Busemann and extended by Van Dyke. The lifting pressure due to a constant vertical acceleration is found, and this is used to calculate the lift coefficient,  $C_{L_a}$ , and the moment coefficient,  $C_{m_a}$ , due to a constant vertical acceleration.

The airfoil's second order contribution to the damping of longitudinal oscillations in aircraft is considered and its relation to the damping of oscillating airfoils is investigated.

## INTRODUCTION

The development of the linearized theory of supersonic flow has permitted a first order evaluation of a number of stability derivatives. Second order theories similar to the one introduced by Busemann (ref. 1) and extended by Van Dyke (refs. 2 and 3) offer possibilities of obtaining second order evaluations of certain stability derivatives, such as the lift,  $C_{L_\alpha}$ , and the moment,  $C_{m_\alpha}$ , due to a constant vertical acceleration.

Examples of the use of second order theories to evaluate stability derivatives are found in refs. (4) and (5).

Recently two papers have appeared (refs. (6) and (7)) dealing with two dimensional oscillating airfoils. The flow over oscillating airfoils is of great importance in flutter calculations. For stability studies, however, it is more convenient to know the flow over airfoils with a constant vertical acceleration and a constant rate of pitch.

In this paper a second order theory is developed for two dimensional airfoils with constant vertical acceleration at supersonic speeds. From this theory the lifting pressure due to constant vertical acceleration is evaluated. The expression for the lifting pressure enables the stability derivatives  $C_{L_\alpha}$  and  $C_{m_\alpha}$  to be calculated. The airfoils con-

sidered herein have arbitrary symmetrical cross sections; however, the analysis can easily be extended to include airfoils with unsymmetrical cross sections.

The flow around an accelerating airfoil is of an unsteady nature and this necessitates the use of time dependent partial differential equations. The differential equation studied here is not the time dependent equation commonly used, since the coordinate axes employed in the analysis are attached to the airfoil.

## ANALYSIS

Introduction: Recent work by Milton D. Van Dyke (refs. 2 and 3) indicates that second order solutions of the partial differential equation of steady supersonic flow can be obtained by iterative methods. In the present paper it will be assumed that iterative methods can also be used to obtain second order solutions of the differential equation for unsteady supersonic flow. This assumption was made in ref. (6) in treating oscillating airfoils.

In addition it will be assumed that the characteristics are the same for the first and second order solutions. This assumption does not appear unreasonable since for steady plane flow the second order solution (ref. (2)) found by an iterative method yields the correct second order pressure of the Busemann second order theory.

The Partial Differential Equation: The partial differential equation for the second order potential flow around an airfoil is (from ref. (4))

$$-\beta^2 \Phi_{x'x'} + \Phi_{z'z'} - \frac{2M^2}{V} \Phi_{x't} - \frac{1}{c^2} \Phi_{tt} + M^2 \left\{ (\gamma-1) (\Phi_{x'} + \frac{1}{V} \Phi_t) (\Phi_{x'x'} + \Phi_{z'z'}) + 2\Phi_{x'} \Phi_{x'x'} + 2\Phi_z \Phi_{xz} + \frac{2}{V} \Phi_x \Phi_{xt} + \frac{2}{V} \Phi_z \Phi_{zt} \right\} \quad (1)$$

The perturbation potential function,  $\Phi$ , has been normalized through division by the free stream velocity. Also from ref. (4) the pressure relation is

$$C_p = -2 \left( \Phi_x + \Phi_t/V \right) \Phi_z - \beta^2 \Phi_x^2 + 2M^2 \Phi_x \Phi_t/V + M^2 \Phi_t^2/V^2 \quad (2)$$

These two equations are associated with axes fixed in space. For the problem considered here it is convenient to express the differential equation in terms of a new set of axes fixed to the airfoil. The relations between the two sets of axes are (see fig. 1)

$$\begin{aligned} x &= x \\ z &= z + \frac{\dot{a}V(t)}{2} \\ t &= t \end{aligned} \quad (3)$$

For the axes attached to the airfoil eqs. (1) and (2) become

$$\begin{aligned} & -\beta^2 \Phi_{xx} + \Phi_{zz} - (2M^2 \Phi_{xt}/V) - (\Phi_{tt}/c^2) + 2M^2 \left\{ \dot{a}t \Phi_{xz} + \right. \\ & \quad \left. (\dot{a}t \Phi_{zt}/V) + \left[ \dot{a} \Phi_z / (2V) \right] + \left[ (\gamma-1)/2 \right] (\Phi_x + \Phi_t/V) (\Phi_{xx} + \Phi_{zz}) + \right. \\ & \quad \left. \Phi_x \Phi_{xx} + \Phi_z \Phi_{xz} + (\Phi_x \Phi_{xt}/V) + (\Phi_z \Phi_{zt}/V) \right\} \end{aligned} \quad (4)$$

$$\begin{aligned} C_p = & -2 \left[ \Phi_x + (\Phi_t/V) + \dot{a}t \Phi_z \right] \Phi_z - \beta^2 \Phi_x^2 + (2M^2 \Phi_x \Phi_t/V) + \\ & (M^2 \Phi_t^2/V^2) \end{aligned} \quad (5)$$

Eqs. (4) and (5) will be used in the analysis of accelerating airfoils.

Solution By Iteration: On the assumption that eq. (4) can be solved by an iterative procedure the initial step will be finding the first order solution. The first order partial differential equation,



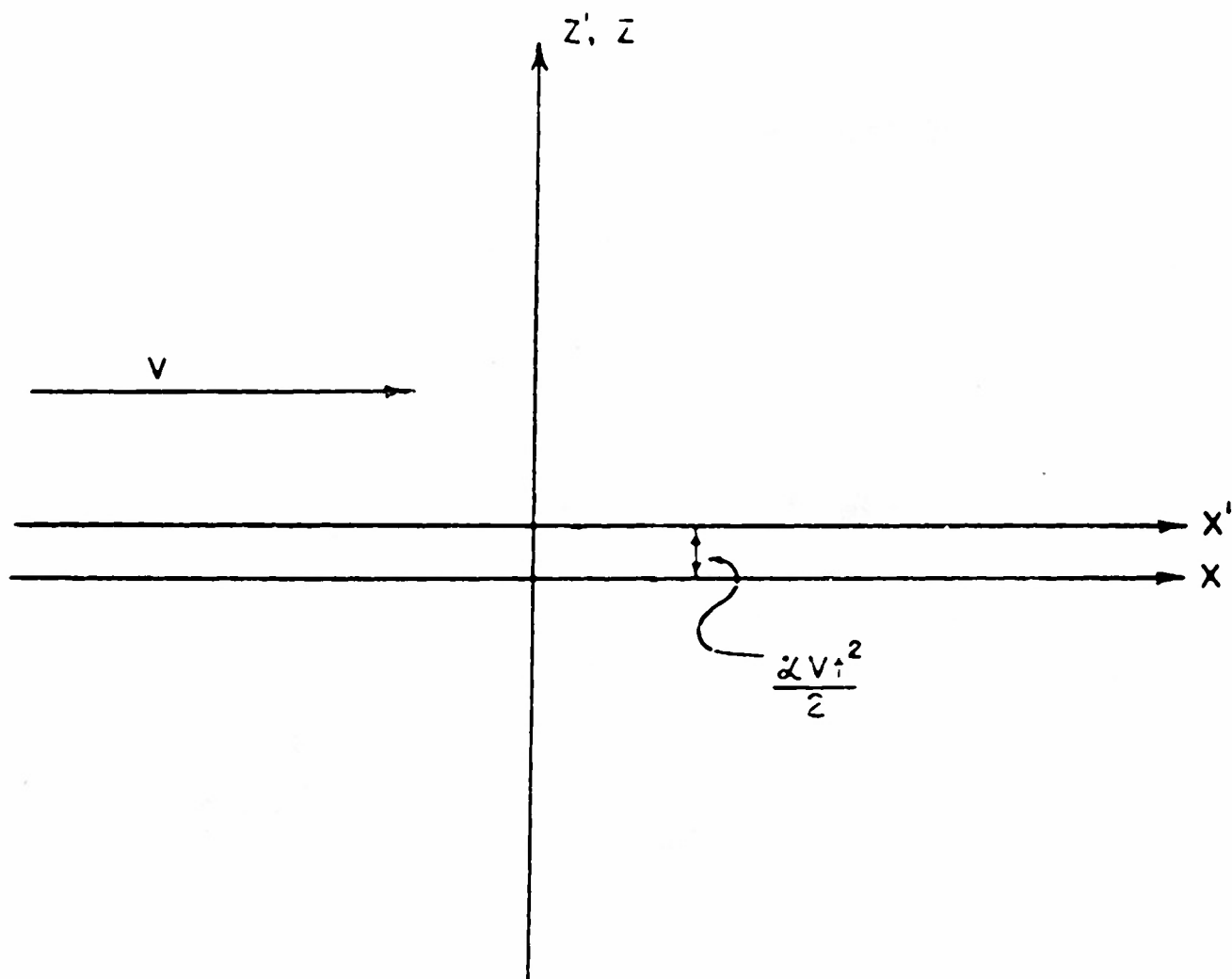


Fig. 1

obtained by neglecting the second order terms in eq. (4), is

$$-\beta^2 \phi_{xx} + \phi_{zz} - (2M^2 \phi_{xt}/V) - (\phi_{tt}/c^2) = 0 \quad (6)$$

The solution to eq. (6) is taken as the first approximation to the solution of eq. (4). We shall assume that the second approximation can be found by substituting the first order solution into the right side of eq. (4) and solving the resulting non-homogeneous equation, which from eqs. (4) and (6), is

$$-\beta^2 \psi_{xx} + \psi_{zz} - (2M^2 \psi_{xt}/V) - \psi_{tt}/c^2 = 2M^2 \left\{ \dot{a}t \phi_{xz} + (\dot{a}t \phi_{zt}/V) + \left[ \dot{a} \phi_z/(2V) \right] - \left[ (\gamma-1)/2 \right] (\phi_x + \phi_t/V)(\phi_{xx} + \phi_{zz}) + \phi_x \phi_{xx} + \phi_x \phi_{xz} + (\phi_x \phi_{xt}/V) + (\phi_z \phi_{zt}/V) \right\} \quad (7)$$

The solution of eq. (7) will be referred to as the second order solution.

Examination of Form of Potential Function: It is helpful to investigate the type of solution obtained from eq. (7). The first order solution will be of the form  $\phi = \alpha g_1(x, z) + \dot{a} g_2(x, z, t) + \epsilon g_3(x, z)$ , where  $\epsilon$  is a thickness parameter and where  $\alpha$ ,  $\dot{a}$ , and  $\epsilon$  are small compared to unity. It follows from the above expression and eq. (7) that the second order solution will be of the form

$$\psi = \alpha^2 h_1(x, z) + \alpha \dot{a} h_2(x, z, t) + \dot{a}^2 h_3(x, z, t) + \alpha \epsilon h_4(x, z) + \dot{a} \epsilon h_5(x, z, t) + \epsilon^2 h_6(x, z)$$

This paper is concerned only with the lifting pressure and its integrated effects. The terms  $\alpha^2 h_1(x, z)$ ,  $\alpha \dot{a} h_2(x, z, t)$ , and  $\dot{a}^2 h_3(x, z, t)$  contribute nothing to the lifting pressure. This can be seen from the following argument. The thickness parameter,  $\epsilon$ , is not present; therefore, the airfoils can be considered to have zero thickness when these terms alone are treated. If the potential of the flow on the upper surface of a flat plate is expressed as

$$\Phi(\alpha, \dot{a}) = \alpha g_1 + \dot{a} g_2 + \alpha^2 h_1 + \alpha \dot{a} h_2 + \dot{a}^2 h_3$$

the potential on the lower surface is given by

$$\Phi(\alpha, -\dot{a}) = -\alpha g_1 - \dot{a} g_2 + \alpha^2 h_1 + \alpha \dot{a} h_2 + \dot{a}^2 h_3$$

The potential difference is

$$\Delta \Phi = 2\alpha g_1 + 2\dot{a} g_2$$

Since for the flat plate the pressure difference between the upper and lower surfaces can be found directly from the potential difference, the terms  $\alpha^2 h_1$ ,  $\alpha \dot{\alpha} h_2$ , and  $\ddot{\alpha}^2 h_3$  do not affect the lifting pressure.

The terms on the right side of eq. (7) which are multiplied by  $\alpha^2$ ,  $\alpha \dot{\alpha}$ , and  $\ddot{\alpha}^2$  can be neglected; for eq. (7) is linear, and  $\alpha^2 h_1$ ,  $\alpha \dot{\alpha} h_2$ , and  $\ddot{\alpha}^2 h_3$  do not contribute to the lifting pressure. The terms  $\alpha \epsilon h_4$  and  $\epsilon^2 h_6$  are independent of  $\dot{\alpha}$  (the acceleration parameter), and hence they also contribute nothing to the pressure due to acceleration.\*

The remaining term  $\dot{\alpha} \epsilon h_5(x, z, t)$  will be found by use of eq. (7), neglecting all expressions involving  $\alpha^2$ ,  $\alpha \dot{\alpha}$ ,  $\ddot{\alpha}^2$ , and  $\alpha \ddot{\alpha}$ .

A further consideration of the form of the solution indicates that the second order lifting pressure is linear in the thickness parameter. This can be established by considering a first order solution of the form

$$\phi = \alpha g_1(x, z) + \dot{\alpha} g_2(x, z, t) + \epsilon_1 g_4(x, z) + \epsilon_2 g_5(x, z)$$

Then the second order lifting pressure will be of the form

$$\Delta C_p = I_0(x, t) + \dot{\alpha} \epsilon_1 I_1(x, t) + \dot{\alpha} \epsilon_2 I_2(x, t) + \dots$$

This equation is linear in  $\epsilon_1$  and  $\epsilon_2$ , thus the lifting pressure for various known thickness distributions can be added to obtain the lifting pressure for other thickness distributions.

This linearity has no special value for the two dimensional airfoil, since we shall determine the solution for an arbitrary thickness distribution. It can be shown similarly that linearity of lifting pressure with thickness distribution holds for three dimensional airfoils also, and for this case it should prove quite useful.

Boundary Conditions: Physical considerations require that the flow be tangent to the surface of the airfoil and that all velocity perturbations vanish upstream of the airfoil. These boundary conditions may be expressed mathematically as:

$$\phi(x, y, z, t) = 0 \quad \text{upstream of the airfoil.}$$

$$\psi(x, y, z, t) = 0$$

and

$$v = \nabla \psi = 0$$

---

\* The terms  $\alpha \epsilon h_4$ ,  $\epsilon^2 h_6$ , and  $\alpha^2 h_1$  are associated with steady supersonic flow and can be found by the Busemann second order theory.

where  $s(x, z) = 0$  is the equation of the surface of the airfoil.

The equation of the surface of the airfoil may also be expressed as

Thus

$$\nabla_s = -1 \epsilon f' + k$$

Since the velocity vector  $v$  can be written as

$$v = 1 (1 + \phi_x + \psi_x) + \lambda (\phi_z + \psi_z + \dot{a}t)$$

it follows that the boundary condition on the airfoil surface is

$$(1 + \phi_x + \psi_x) \epsilon f' + \phi_z + \psi_z + \dot{a}t = 0$$

The coordinate axes will be chosen so that the airfoil lies approximately in the  $z = 0$  plane. The boundary conditions for the first order solution are  $\phi(x, y, z) = 0$  upstream of the airfoil, and

$$\phi_z \Big|_{z=0} = -\dot{a} \quad \dot{a}t = \epsilon f \quad (8)$$

Since only the  $\dot{a}t$  term is being calculated  $\dot{a}$  will be set equal to zero. Similarly the boundary conditions for the second order solution are  $\psi = 0$  upstream of the airfoil, and

$$\psi_z \Big|_{z=0} = -\phi_x \Big|_{z=0} \quad f = \epsilon f \quad \phi_{zz} \Big|_{z=0} \quad (9)$$

For the airfoils considered in this paper the first order velocity components are discontinuous across the Mach sheet from the leading edge. The effect of these discontinuities on the second order solution will be determined by assuming that a small transition surface is attached to the leading edge. This small surface is so shaped that the discontinuities in the velocity components are removed. The effect of the discontinuities will be evaluated by a limiting process in which the width of the attached surface approaches zero. A surface which will remove the first order velocity discontinuities in the flow over the upper surface is

$$z = (\sigma x^2 t / 2) + (x^2 a / 2)$$

where the surface extends from  $x = 0$  to  $x = l$ , and  $\sigma l = \dot{a}$ ,  $z l = \epsilon$ .

Evaluating the second order potential function associated with the flow over the small transition surface introduced above will lead to the determination of the second order leading edge discontinuities. The first order solution for the flow over the transition surface is

$$\phi = \left[ (x - \beta z)^2 / (2\beta V) \right] \left[ -\left[ \sigma M^2 (x + 2\beta z) \right] / (3\beta^2) + \sigma V t + V a \right] \quad (10)$$

The second order solution can be found by substituting eq. (10) into eq. (7) and solving the resulting non-homogeneous equation; however, in this case it is easier to use a different approach. In appendix A it is shown that the solution for planar problems to the three dimensional non-homogeneous equation

$$-\beta^2 \psi_{xx} + \psi_{yy} + \psi_{zz} - (2M^2 \psi_{xt}/V) - (\psi_{tt}/c^2) = F(x, y, z, t) \quad (11)$$

can be written as

$$\psi(x, y, z, t) = \frac{1}{4\pi} \iiint_{V_0} \frac{1}{R} [F(t_1) + F(t_2)] dv + \frac{1}{2\pi} \iint_{S_0} \frac{1}{R} \left\{ \left[ \frac{\partial \psi(\xi, \eta, 0, t_1)}{\partial \zeta} \right] + \left[ \frac{\partial \psi(\xi, \eta, 0, t_2)}{\partial \zeta} \right] \right\} da \quad (12)$$

where  $V_0$  is the volume inclosed by the forward Mach cone from the point  $(x, y, z)$ , and the surface  $S_0$  is the area of the transition surface in the forward Mach cone from the point  $(x, y, z)$ .

When eq. (12) is applied to an arbitrary point located on the Mach surface from the trailing edge of the transition surface, the result can be expressed as

$$\begin{aligned} \psi(x, z, t) = & \frac{1}{4\pi} \int_0^z d\zeta \int_{\beta\zeta}^{\beta\zeta + l} d\xi \left\{ \frac{(\sqrt{(x-\xi)^2 - \beta^2(z-\zeta)^2})/\beta}{-(\sqrt{(x-\xi)^2 - \beta^2(z-\zeta)^2})/\beta} \right. \\ & \left. \frac{1}{2\pi} \int_0^l d\xi \left\{ \frac{(\sqrt{(x-\xi)^2 - \beta^2 z^2})/\beta}{-(\sqrt{(x-\xi)^2 - \beta^2 z^2})/\beta} \right\} \frac{1}{R} \left\{ \left[ \frac{\partial \psi(\xi, 0, 0, t_1)}{\partial \zeta} + \left[ \frac{\partial \psi(\xi, 0, 0, t_2)}{\partial \zeta} \right] \right\} d\eta \right. \right. \\ & \left. \left. \frac{1}{R} [F(t_1) + F(t_2)] d\eta \right\} \right. \end{aligned}$$

Since  $F$ ,  $[\partial \psi(\xi, 0, 0, t_1)/\partial \zeta]$ , and  $[\partial \psi(\xi, 0, 0, t_2)/\partial \zeta]$  are independent of  $\eta$ , the preceding equation can be reduced to

$$\begin{aligned} \psi(x, z, t) = & \frac{1}{4} \int_0^z d\zeta \int_{\beta\zeta}^{\beta\zeta + l} [F(\xi, \zeta, t_1) + F(\xi, \zeta, t_2)] d\xi + \\ & \frac{1}{2} \int_0^l \left\{ \left[ \frac{\partial \psi(\xi, 0, 0, t_1)}{\partial \zeta} \right] + \left[ \frac{\partial \psi(\xi, 0, 0, t_2)}{\partial \zeta} \right] \right\} d\xi \quad (13) \end{aligned}$$

The integrand of the first integral in eq. (13) can be written as

$$F(\xi, \zeta, t_1) + F(\xi, \zeta, t_2) = J_0(\xi, \zeta, t) + (\xi - \beta \zeta) J_1(\xi, \zeta, t) + (\xi - \beta \zeta)^2 J_2(\xi, \zeta, t) + (\xi - \beta \zeta)^3 J_3(\xi, \zeta, t)$$

where  $J_0, J_1, J_2$ , and  $J_3$  are continuous functions and are not zero at  $\xi = \beta \zeta$ .

Since the  $\xi$  in the integrands of eq. (13) varies over a very small range, the functions  $J_0, J_1, J_2$  and  $J_3$  can be considered as constants with respect to integration in the  $\xi$  direction. Therefore  $\xi$  can be replaced by  $\beta \zeta$  in these integrands, thus permitting integration to be performed with respect to  $\xi$ . Eq. (13) becomes

$$\begin{aligned} \psi(x, z, t) = & \frac{\ell}{4\beta} \int_0^z J_0(\beta \zeta, \zeta, t) d\zeta + \frac{\ell^2}{8\beta} \int_0^z J_1(\beta \zeta, \zeta, t) d\zeta + \\ & \frac{\ell^3}{12\beta} \int_0^z J_2(\beta \zeta, \zeta, t) d\zeta + \frac{\ell^4}{16\beta} \int_0^z J_3(\beta \zeta, \zeta, t) d\zeta + \quad (14) \\ & \frac{\ell^2}{2} \left[ \frac{\dot{\sigma} M^2}{\beta^2} \left( \frac{x}{2} - \frac{\ell}{3} \right) - \frac{\dot{\sigma} V t}{2} + \frac{a}{2} \right] \end{aligned}$$

The value of the second order potential function on the downstream side of the leading edge Mach sheet above the airfoil can be found by taking the limit of eq. (14) as  $\ell$  approaches zero and as  $\dot{\sigma}$  and  $a$  approach infinity so that

$$\begin{aligned} \lim_{\substack{\ell \rightarrow 0 \\ \dot{\sigma} \rightarrow \infty \\ a \rightarrow \infty}} \dot{\sigma} \ell &= \dot{a} \end{aligned}$$

and

$$\begin{aligned} \lim_{\substack{\ell \rightarrow 0 \\ a \rightarrow \infty}} a \ell &= \epsilon \end{aligned}$$

Thus the discontinuity in the second order potential function is given by

$$\psi'_{x=\beta z} = \lim_{\substack{\ell \rightarrow 0 \\ \dot{\sigma} \rightarrow \infty \\ a \rightarrow \infty}} \frac{\ell}{4\beta} \int_0^z J_0(\beta \zeta, \zeta, t) d\zeta + \lim_{\substack{\ell \rightarrow 0 \\ \dot{\sigma} \rightarrow \infty \\ a \rightarrow \infty}} \frac{\ell^2}{8\beta} \int_0^z J_1(\beta \zeta, \zeta, t) d\zeta \quad (15)$$

The integrals containing  $J_2$  and  $J_3$  are zero in the limit since  $J_2$  and  $J_3$

are linear functions of  $\dot{\sigma}^2$ ,  $\dot{\sigma}a$ ,  $a^2$ ,  $\dot{a}^2$ , and  $\dot{a}a$ .

From eqs. (10) and (7) the functions  $J_0$  and  $J_1$  are (neglecting the  $\dot{\sigma}^2$ ,  $\dot{\sigma}\dot{a}$ ,  $a^2$  and  $\dot{a}^2$  terms)

$$J_0 = 4\dot{a} M^2 a \left[ -t + M(x - \beta z) / (\beta^2 c) \right]$$

$$J_1 = -4(\gamma+1)M^4 \dot{\sigma}a \left[ Vt - M^2 x / \beta^2 \right] / (V\beta^2)$$

Substituting the preceding expression into eq. (15) and performing the indicated operations yield

$$\begin{aligned} \psi|_{x=\beta z} &= \dot{a} \epsilon (M^2 / \beta^2) \left[ -t + Mx / (2\beta^2 c) \right] \beta z + \\ &\quad \dot{a} \epsilon \left[ (\gamma+1)/2 \right] (M^4 / \beta^4) \left[ t - Mx / (\beta^2 c) \right] \beta z \end{aligned} \quad (16)$$

Expression (16) is the value of the discontinuity in the second order potential function across the leading edge Mach sheet above the airfoil.

Solution of the Partial Differential Equation: The part of the second order potential function which contributes to the lifting pressure due to a constant acceleration will now be determined. Since the method of solution is essentially the same for both the upper and lower surfaces, only the flow over the upper surface will be considered in detail.

The first order solution is

$$\phi = \frac{\dot{a}}{\beta V} \left[ -\frac{M^2 x^2}{2\beta^2} + \frac{M^2}{2} z^2 + Vt(x - \beta z) \right] - \epsilon \frac{f(x - \beta z)}{\beta} \quad (17)$$

It follows from eqs. (7) and (17) that the second order potential function must satisfy the non-homogeneous equation

$$\begin{aligned} -\beta^2 \psi_{xx} + \psi_{zz} - \frac{2M^2}{V} \psi_{xt} - \frac{1}{c^2} \psi_{tt} &= \frac{M^2 \dot{a} \epsilon}{V} \left\{ \frac{1 + \gamma M^2 (1 - \beta^2)}{\beta^4} f' - \right. \\ &\quad \left. \left[ (\gamma M^2 - M^2 + 2) / \beta^2 \right] Vt f'' + (M^2 / \beta^4) (\gamma + 1) (x - \beta z) f'' \right\} \end{aligned} \quad (18)$$

where the  $\dot{a}^2$ ,  $\dot{a}\dot{a}$ ,  $\dot{a}\epsilon$ ,  $\epsilon^2$ , and  $\dot{a}^2$  terms have been neglected.

The potential function  $\psi$  will be divided into two parts  $\psi_1$  and  $\psi_2$  such that  $\psi_1$  satisfies the non-homogeneous equation and  $\psi_2$  satisfies the homogeneous equation.

By inspection a solution of the non-homogeneous equation is found to be

$$\theta_1 = -\frac{\dot{a} \epsilon M^2}{2\beta^3} (M^2 \gamma - M^2 + 2) z t f' - \frac{\dot{a} \epsilon M^2}{4V \beta^5} [\gamma(\beta^4 - 2) + M^4 - 3] z f - \frac{\dot{a} \epsilon M^2}{4V \beta^4} [\beta^4 - \gamma(M^4 - 1)] z^2 f' + \frac{\dot{a} \epsilon M^4}{2V \beta^5} (\gamma + 1) x z f' \quad (19)$$

The boundary conditions are given by eqs. (9) and (16). From eq. (19) and the boundary conditions it follows that

$$\theta_2 \Big|_{x=\beta z} = 0$$

and

$$\frac{\partial \theta_2}{\partial z} \Big|_{z=0} = \frac{\dot{a} \epsilon}{2V \beta^3} \left[ - (M^4 \gamma - M^4 + 2) V t f' - \frac{M^2}{2\beta^2} (-2M^2 \gamma + 4\beta^4 - 2) f + (M^2/\beta^2)(M^2 \gamma - M^2 + 2) x f' \right]$$

By inspection

$$\theta_2 = \frac{\dot{a} \epsilon}{2V \beta^4} \left\{ (M^4 \gamma - M^4 + 2) V t f + (M^2/\beta^2) \left[ -M^2 (\gamma - 1) \beta^2 z - (M^2 \gamma - M^2 + 2) x \right] f + \frac{M^2}{2\beta^2} (-2M^4 \gamma + 6M^4 - 10M^2 + 2) \int_0^{x - \beta z} f(u) du \right\} \quad (20)$$

Thus, from eqs. (19) and (20), the part of the second order potential function which contributes to the lifting pressure is

$$\psi = \frac{\dot{a} \epsilon}{2V \beta^4} \left\{ M^2 (M^2 \gamma - M^2 + 2) \beta z V t f' - \gamma M^4 \beta^2 z^2 f' - (M^4/\beta^2)(\gamma + 1) x \beta f' z + (M^4 \gamma - M^4 + 2) V t f - (M^2/\beta^2) \left[ (M^4 \gamma - M^4 + M^2 + 1) \beta z + (M^2 \gamma - M^2 + 2) x \right] f - (M^2/\beta^2)(M^4 \gamma - 3M^4 + 5M^2 - 1) \int_0^{x - \beta z} f(u) du \right\} \quad (21)$$

Lifting Pressure: The lifting pressure distribution can be expressed as

$$\Delta C_p = -C_p \Big|_{\text{upper surface}} + C_p \Big|_{\text{lower surface}}$$



From eqs. (2) and (21) it follows that for symmetrical airfoils the lifting pressure distribution at  $t = 0$  is

$$\Delta C_p = \frac{\dot{a} \sqrt{x}}{V \beta^3} + \frac{2\dot{a} \epsilon M^2}{V \beta^6} \left[ (M^2 \gamma - M^2 + 2) x f'(x) + (2M^2 \gamma - 3M^2 + 5) f(x) \right] \quad (22)$$

Fig. 2 presents the lifting pressure distribution at  $t=0$  on a ten percent thick wedge for various Mach numbers, and fig. 3 presents the lifting pressure distribution at  $t = 0$  on a five percent thick airfoil with a parabolic cross section for various Mach numbers.

The Force and Moment: The effect of thickness on the lift,  $C_{L_{\dot{a}}}$ , and the moment  $C_{m_{\dot{a}}}$ , due to a constant vertical acceleration can be found by use of eq. (22). The stability derivatives  $C_{L_{\dot{a}}}$  and  $C_{m_{\dot{a}}}$  can be expressed as

$$C_{L_{\dot{a}}} = \frac{\partial C_L}{\partial \left( \frac{\dot{a} c_r}{2V} \right)} \Big|_{\dot{a} \rightarrow 0} = \left( \frac{1}{\left( \frac{\dot{a} c_r}{2V} \right)^2} \right) \int_0^{c_r} \Delta C_p dx$$

$$C_{m_{\dot{a}}} = \frac{\partial C_m}{\partial \left( \frac{\dot{a} c_r}{2V} \right)} \Big|_{\dot{a} \rightarrow 0} = \left( \frac{1}{\left( \frac{\dot{a} c_r}{2V} \right)^3} \right) \int_0^{c_r} (x-d) \Delta C_p dx$$

From the preceding relations and eq. (22) the  $C_{L_{\dot{a}}}$  and the  $C_{m_{\dot{a}}}$  of an airfoil are found to be

$$C_{L_{\dot{a}}} = \frac{8}{\beta^3} \left\{ \frac{1}{2} + \frac{\epsilon M^2}{2\beta^3} \left[ (M^2 \gamma - M^2 + 2) \frac{f_{TE}}{c_r} + (M^2 \gamma - 2M^2 + 3) \int_0^{c_r} \frac{f(x)}{c_r^2} dx \right] \right\} \quad (23)$$

and

$$C_{m_{\dot{a}}} = \frac{8}{\beta^3} \left\{ \frac{1}{3} - \frac{d}{2c_r} + \frac{\epsilon M^2}{2\beta^3} \left[ (M^2 \gamma - M^2 + 2)(1-d/c_r) \frac{f_{TE}}{c_r} - (d/c_r)(M^2 \gamma - 2M^2 + 3) \int_0^{c_r} \frac{f(x)}{c_r^2} dx - \beta^2 \int_0^{c_r} \frac{x f(x)}{c_r^3} dx \right] \right\} \quad (24)$$

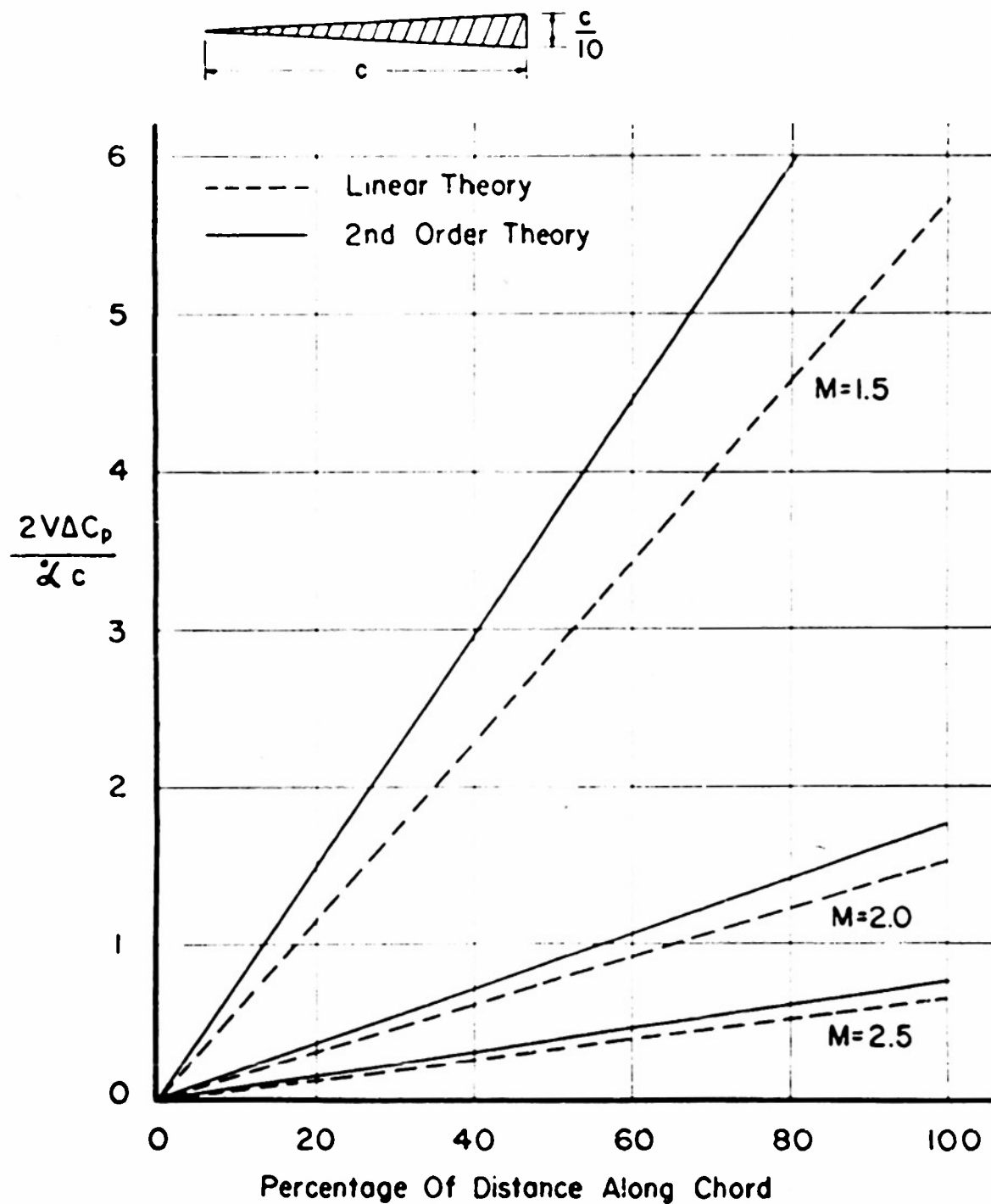


Fig. 2

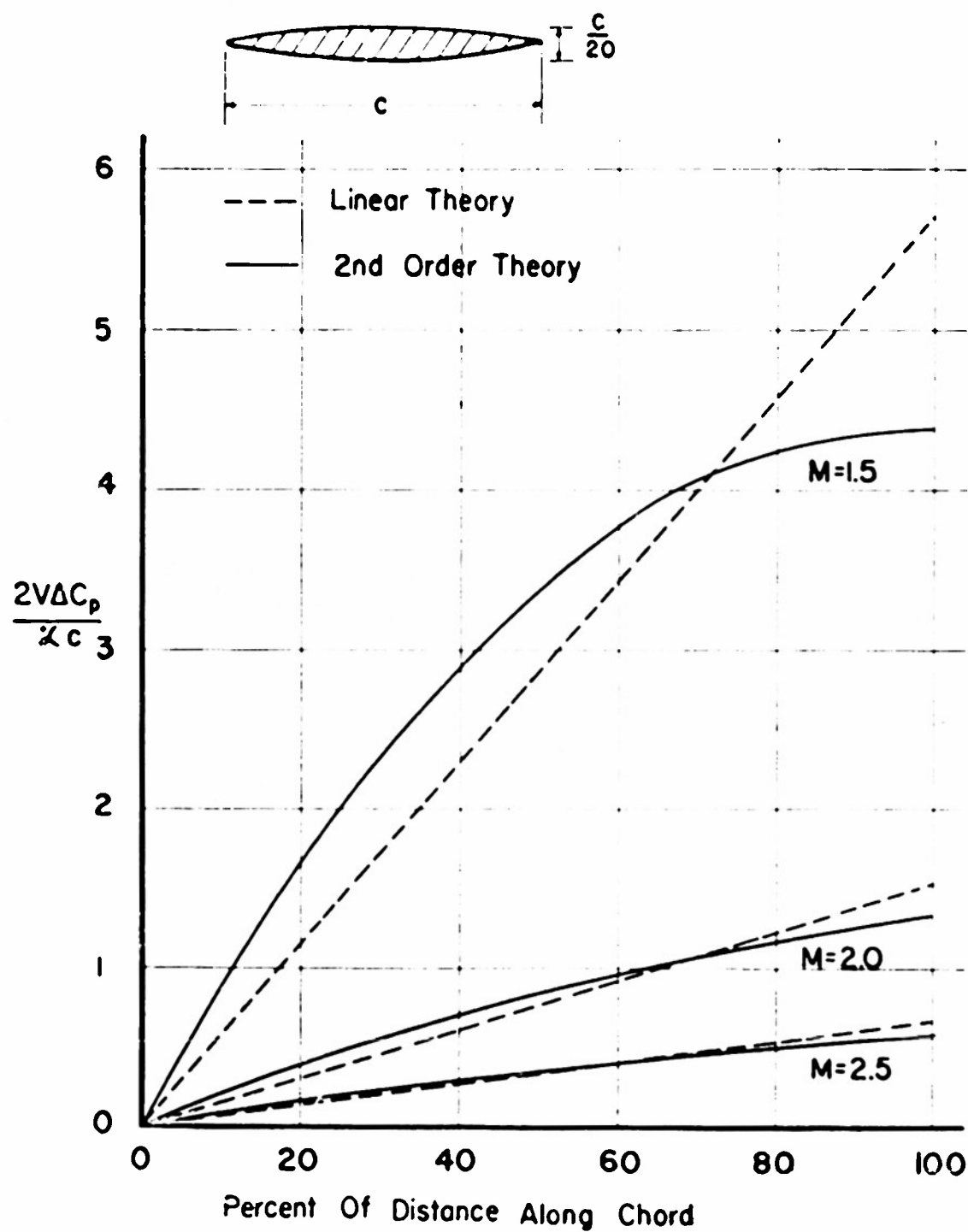


Fig. 3

Fig. 4 presents the variation of the  $C_{L_\alpha}$  of a wedge airfoil with Mach number for various thicknesses; and fig. 5 presents the variation of the  $C_{L_\alpha}$  of an airfoil of parabolic cross section with Mach number for zero and 7.5 percent thicknesses. Fig. 6 presents the variation of the  $C_{m_\alpha}$  of a wedge airfoil with Mach number for various thicknesses, and fig. 7 presents the variation of the  $C_{m_\alpha}$  of an airfoil of parabolic cross section with Mach number for various thicknesses.

The sum of the stability derivatives  $C_{m_\alpha}$  and the damping in pitch,  $C_{m_q}$ , largely determines the damping of the longitudinal oscillations in aircraft. The second order damping in pitch for airfoils with symmetrical thickness distributions is given by eq. (23) of ref. (5). The sum of this equation and eq. (24) can be written as

$$C_{m_\alpha} + C_{m_q} = \frac{8}{\beta} \left[ \frac{1-\beta^2}{3\beta^2} + \frac{(2\beta^2-1)}{2\beta^2} \left( \frac{d}{c_r} \right) - \left( \frac{d}{c_r} \right)^2 \right] + \frac{4\epsilon}{\beta^4} \left\{ \frac{1}{\beta^2} (-M^6 + 4M^4 - 6M^2 + 4 + \gamma M^2) + \right. \\ \left. \left( \frac{d}{c_r} \right) (M^2 + \gamma M^4 - 4M^2 + 4) \right] \left( 1 - \frac{d}{c_r} \right) \frac{f_{TE}}{c_r} - \left( \frac{d}{c_r} \right) \left( \frac{1}{\beta^2} \right) \left[ \gamma M^4 (2M^2 - 1) + 2M^6 - 11M^4 + \right. \\ \left. 18M^2 - 8 \right] \int_0^{c_r} \frac{f(x)}{c_r^2} dx + 2(M^4 + \gamma M^4 - 4M^2 + 4) \int_0^{c_r} \frac{x f(x)}{c_r^3} dx \left. \right\} \quad (25)$$

Fig. 8 presents the variation of the  $C_{m_\alpha} + C_{m_q}$  of a wedge airfoil with Mach number for various thicknesses for the case where the axis of pitch is located at the mid-chord point. Fig. 9 presents the variation of the  $C_{m_\alpha} + C_{m_q}$  of an airfoil with a parabolic cross section with Mach number for various thicknesses for the case where the axis of pitch is located at the midchord point. These two figures indicate that the effect of thickness has a destabilizing effect for a wedge airfoil and has a stabilizing effect for an airfoil with parabolic cross section.

Fig. 10 presents the regions of possible instability for a ten percent wedge and a five percent parabolic arc airfoil. The curves are lines of zero damping found by placing  $C_{m_\alpha} + C_{m_q}$  equal to zero.

This figure indicates that the effect of thickness increases the region of instability for the wedge airfoil and shifts the region to a lower

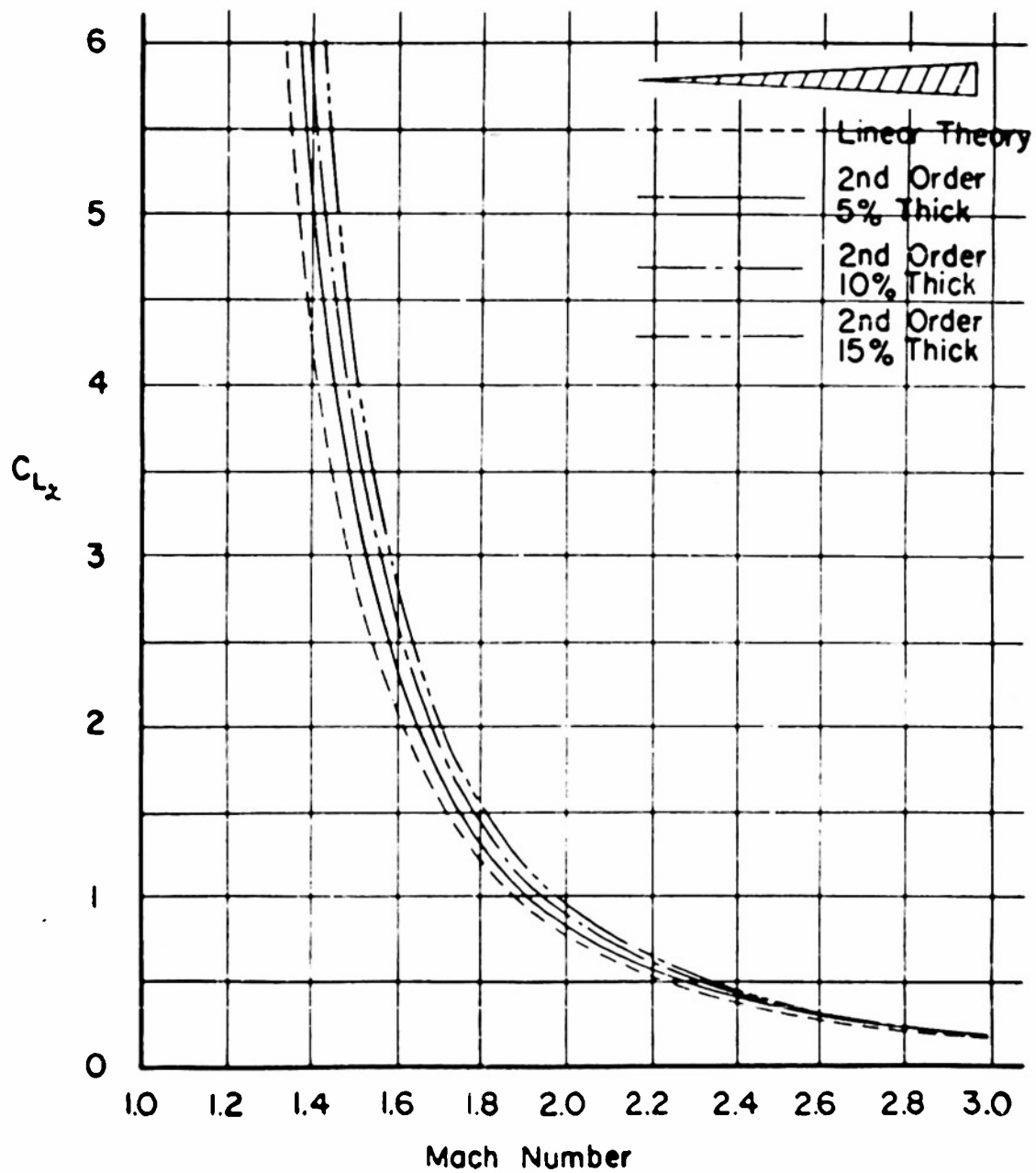


Fig. 4

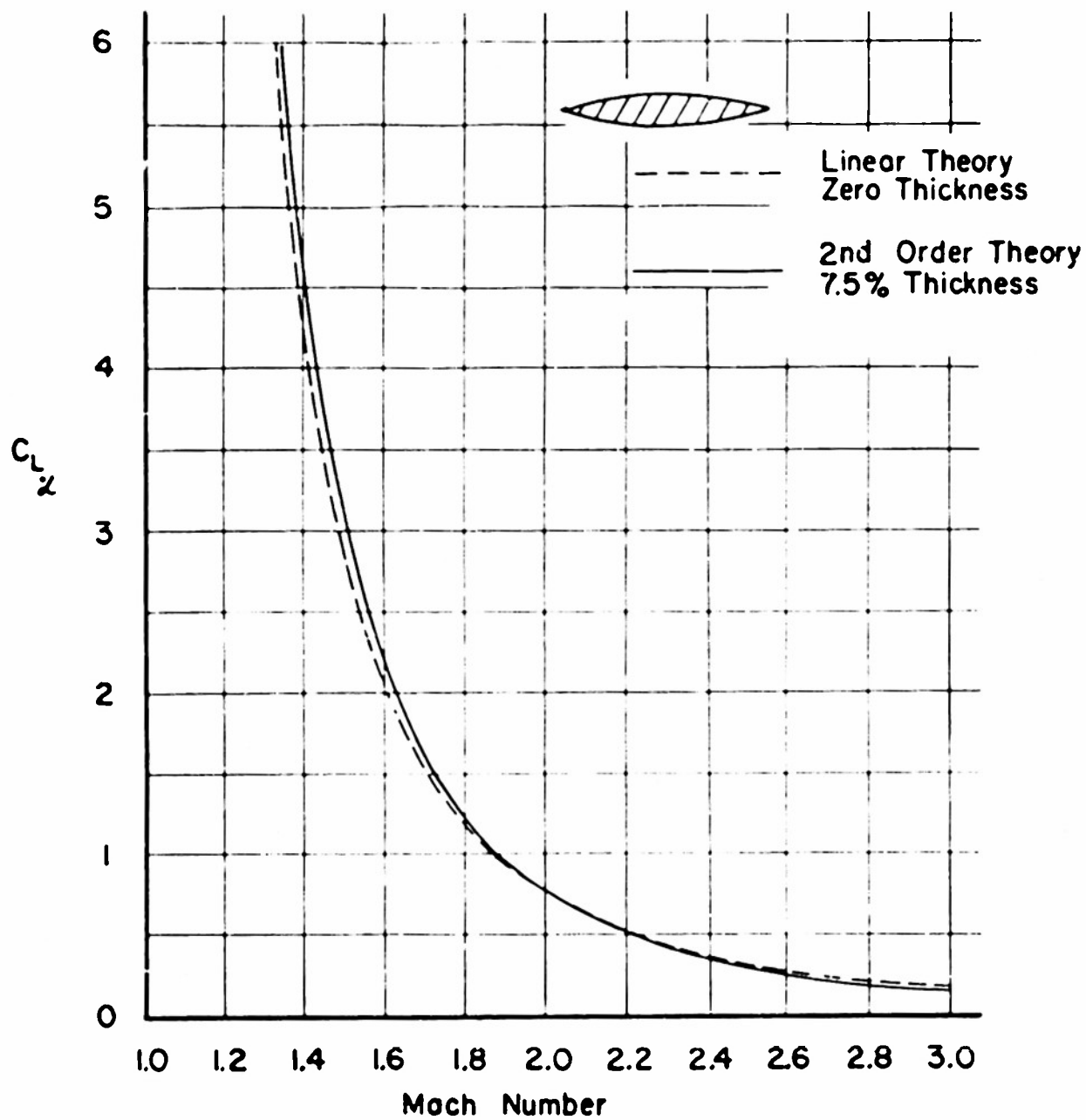


Fig. 5

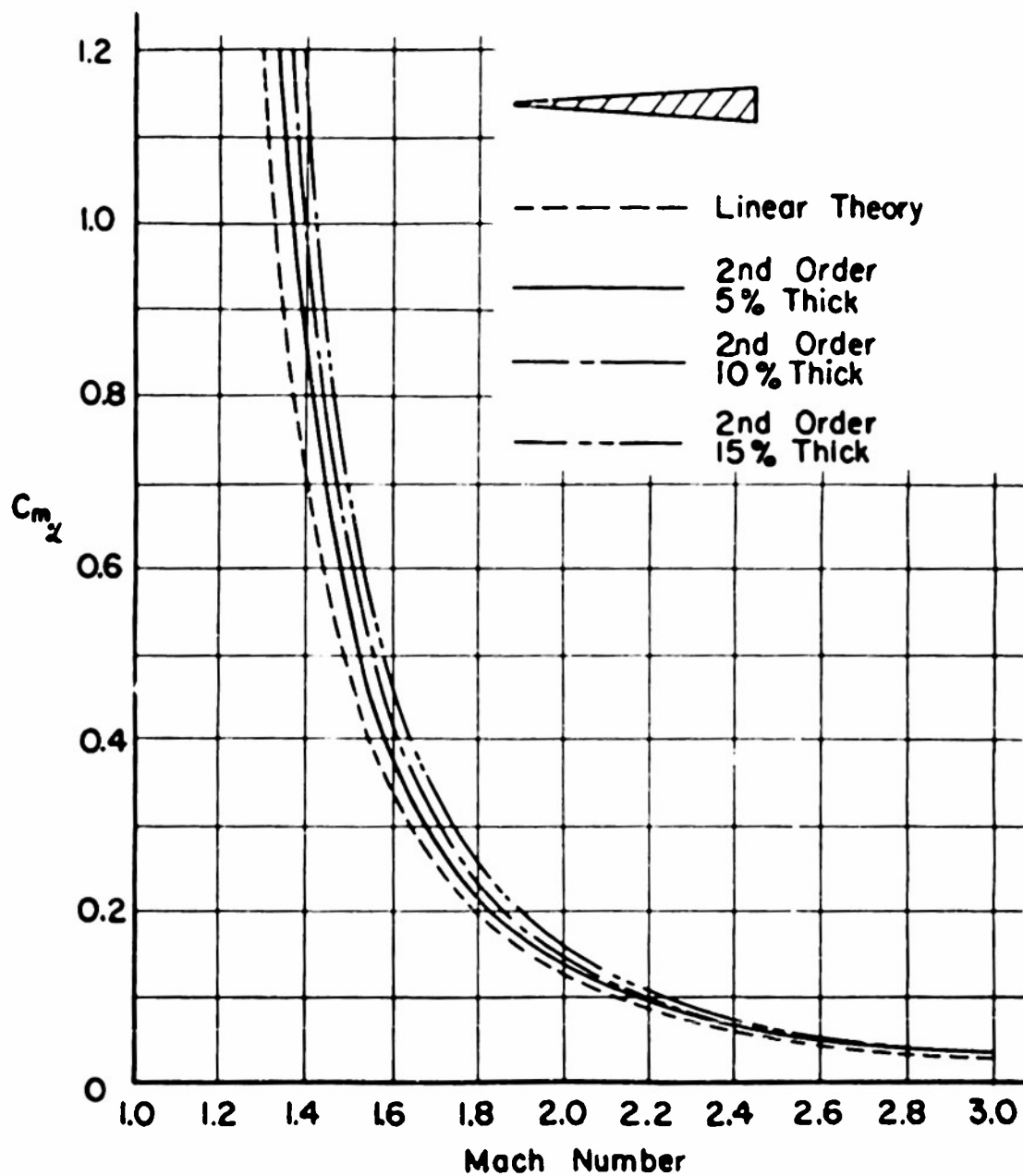


Fig. 6

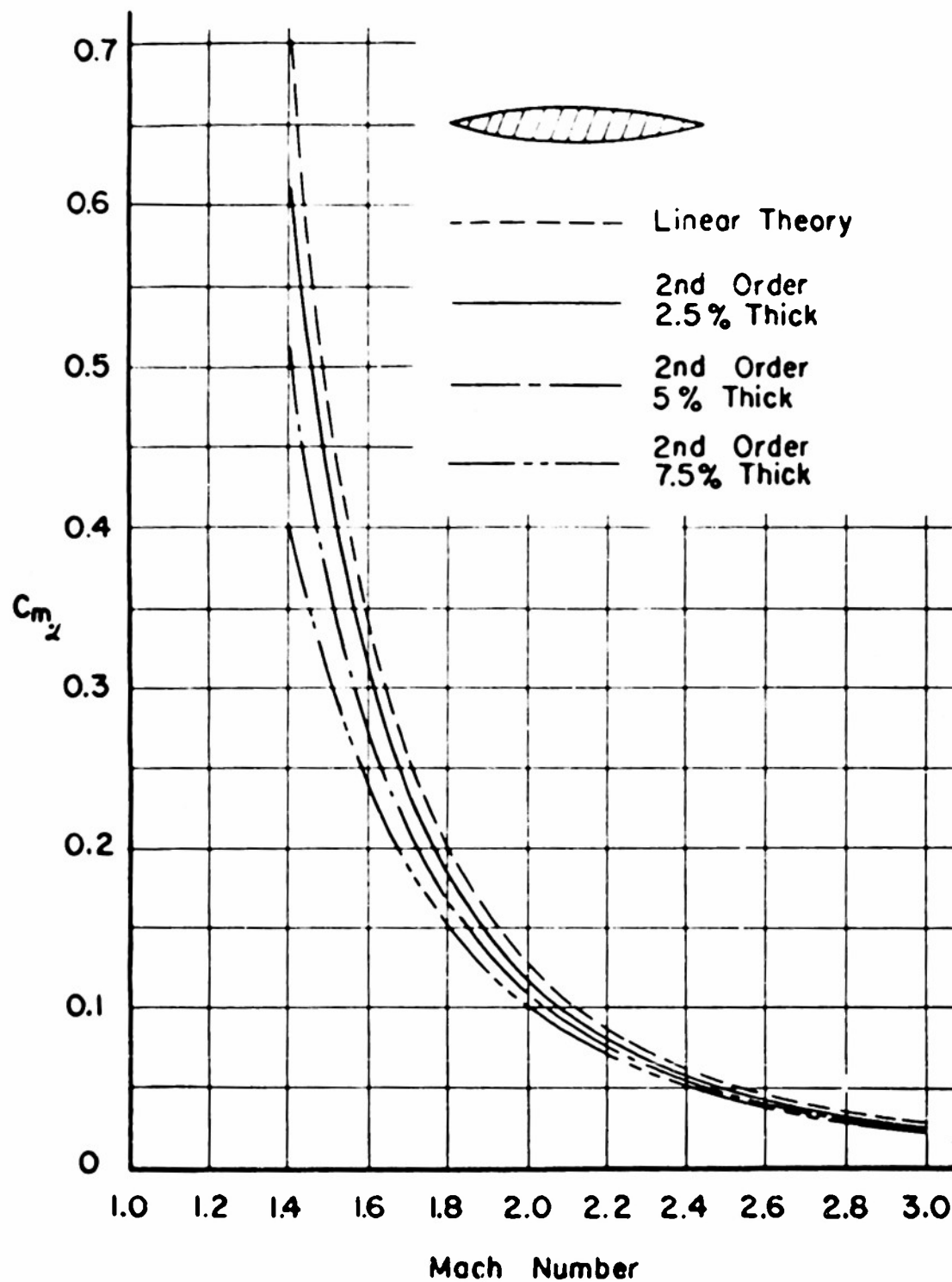


Fig. 7



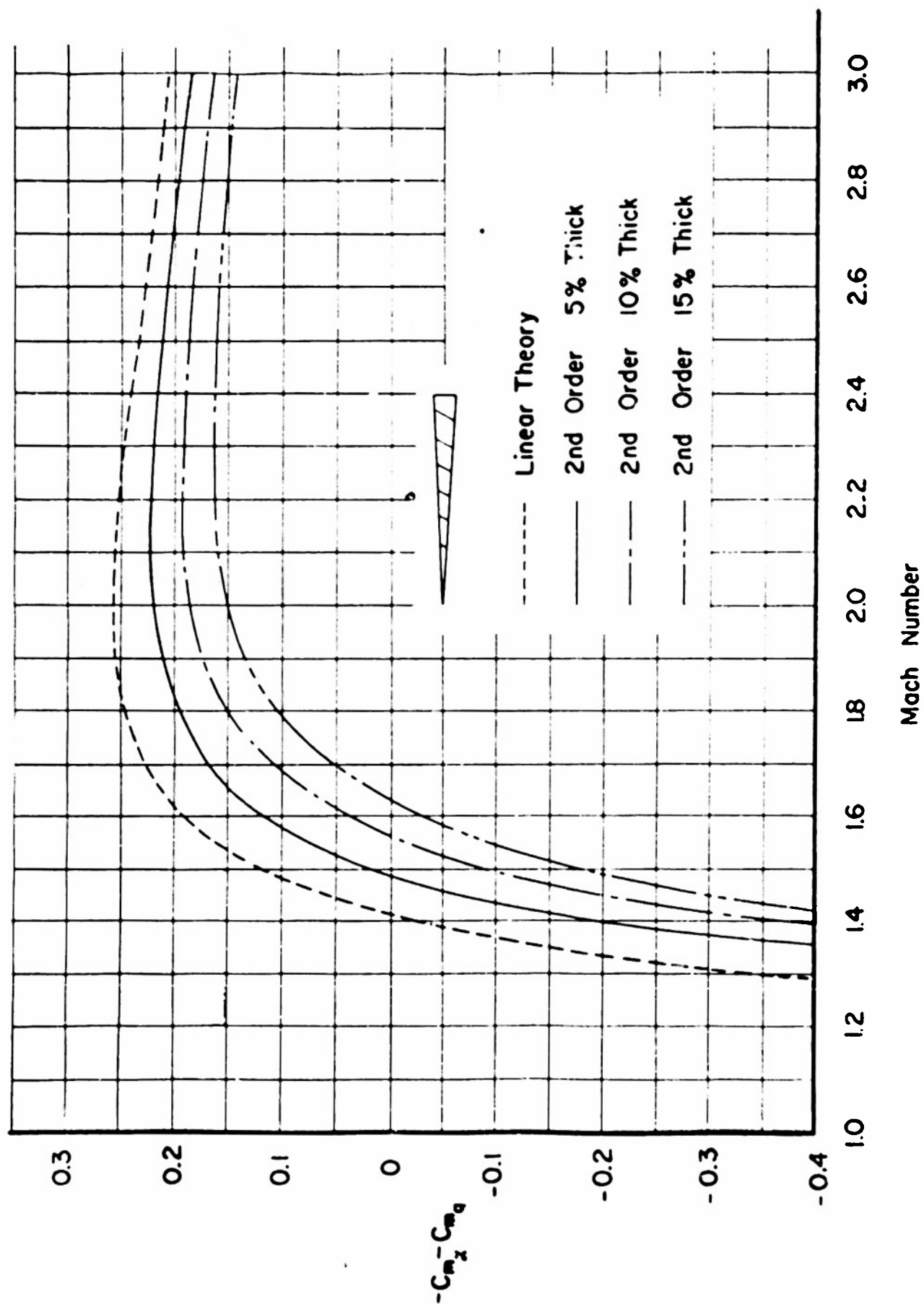


Fig. 8

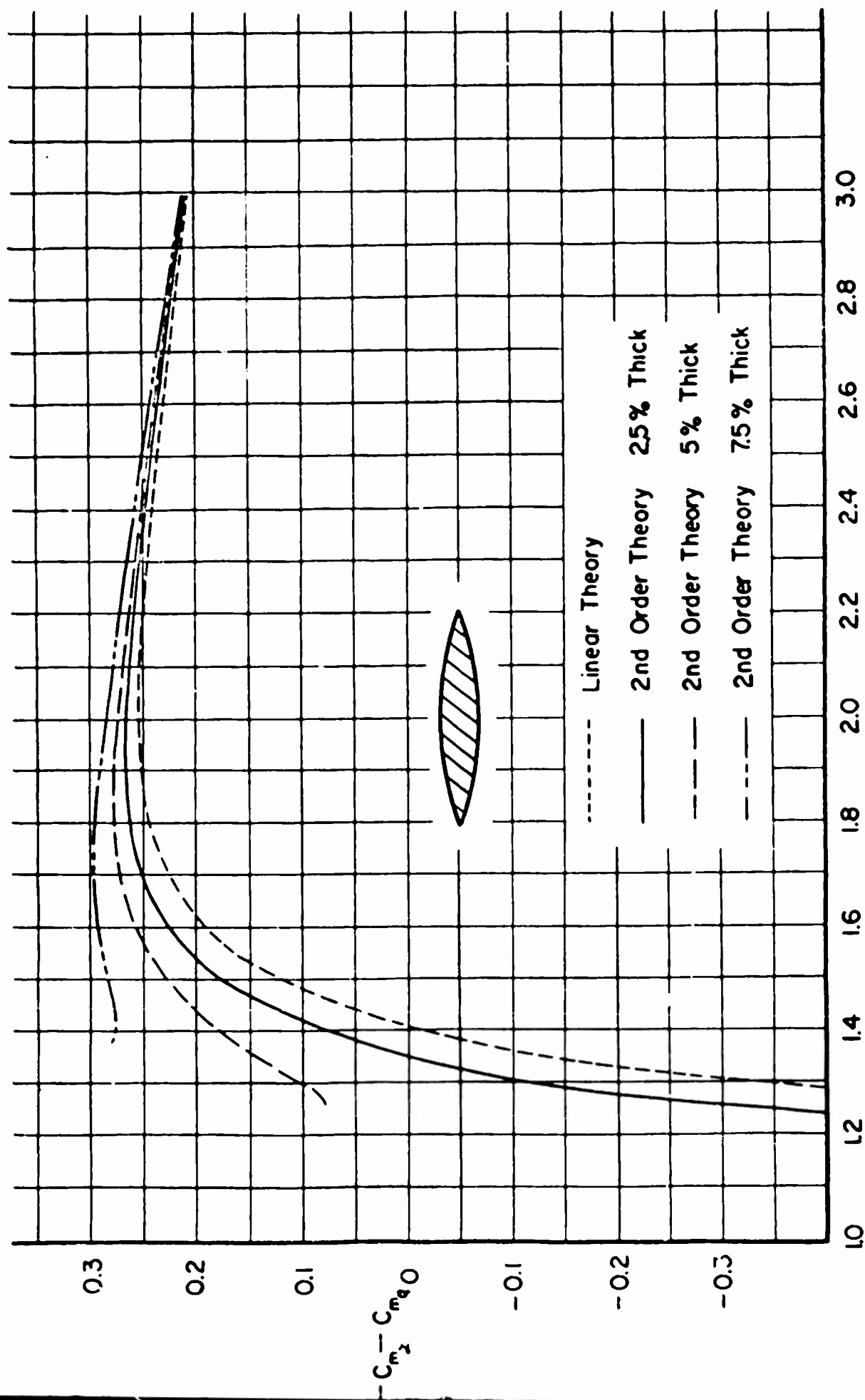


Fig. 9

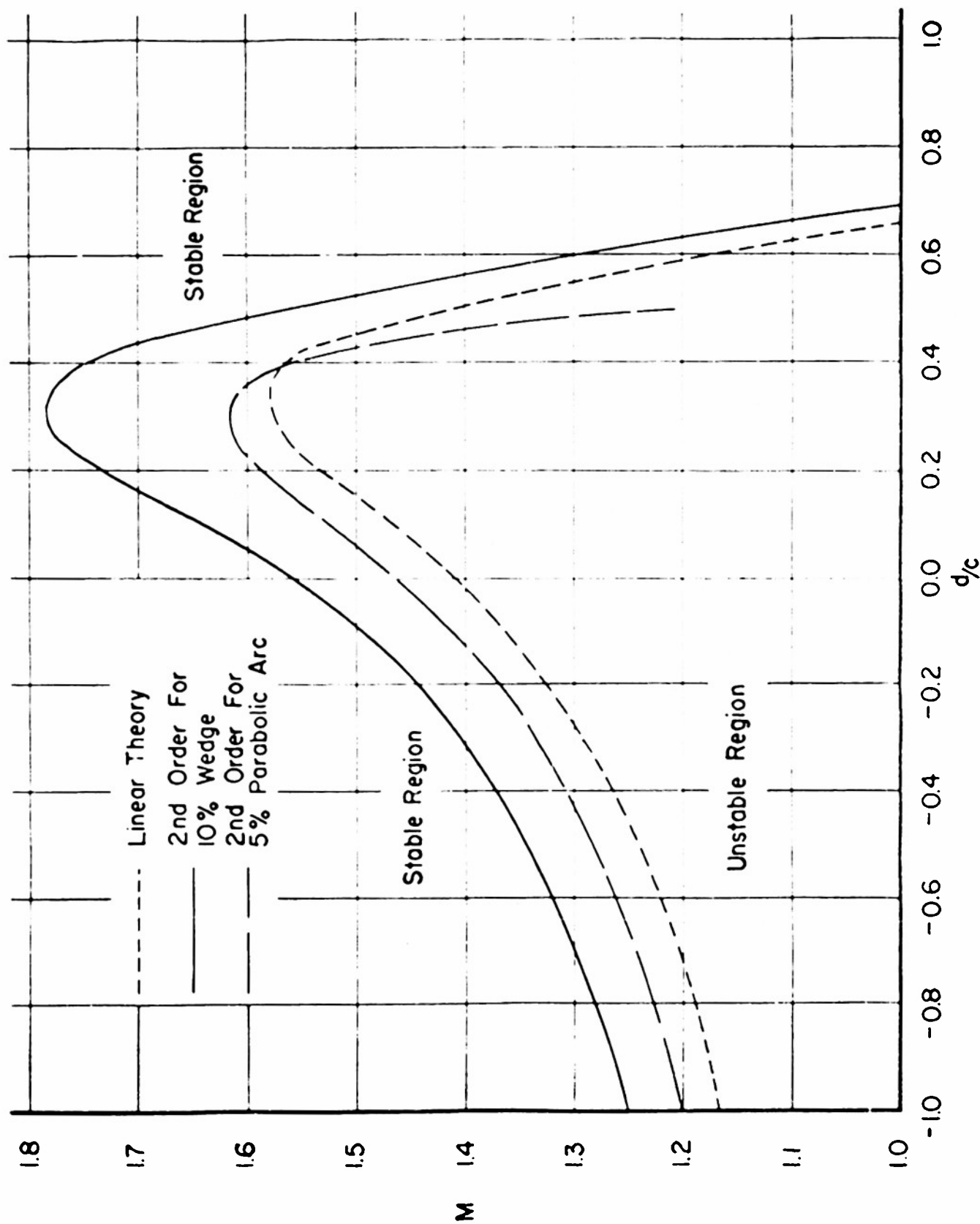


Fig. 10

value of  $d/c_r$  for the parabolic airfoil.

The relation between  $C_{m_\alpha} + C_{m_q}$  and the damping of a slowly oscillating airfoil is investigated in appendix B. It is shown that to the second order in amplitude and thickness, and to the first order in frequency, the damping of a slowly oscillating airfoil is directly proportional to the sum  $C_{m_\alpha} + C_{m_q}$ .

#### CONCLUDING REMARKS

The airfoils considered in this paper have symmetrical thickness distributions. But since the flow over the upper and lower surfaces of the airfoils treated are independent of each other, the aerodynamic properties due to constant vertical accelerations of airfoils with unsymmetrical thickness distributions can easily be determined from the results obtained here.

The limitations of the Busemann second order theory have been investigated (see ref. 8). Since the theory contained in the present paper is closely associated with the Busemann second order theory it seems likely that the results presented herein have similar limitations.

John C. Martin  
JOHN C. MARTIN

Nathan Gerber  
NATHAN GERBER

# REFERENCES

1. Busemann, A.: Aerodynamischer Auftrieb bei Überschallgeschwindigkeit. Atti die convegni 5, R. Accad. d' Italia, 1936, pp 328 - 360 (also printed in Luftfahrtforschung Bd. 12, Nr. 6, Oct 3, 1935. pp 210 - 229; available in translation as British Air Ministry Translation No. 2844.)
2. Van Dyke, M. D.: A Study of Second Order Theory of Supersonic Flow Past Bodies of Revolution. NACA TN 2200, 1951.
3. Van Dyke, M. D.: First and Second Order Theory of Supersonic Flow Past Bodies of Revolution. Jour. Aero. Sci., Vol 18, No. 3, March 1951, pp 161-179.
4. Martin, John C. and Gerber, Nathan: On the Effect of Thickness on the Damping in Roll of Airfoils at Supersonic Speeds. Ballistic Research Laboratories (Aberdeen Proving Ground, Md.) Report 843.
5. Martin, John C. and Gerber, Nathan: The Effect of Thickness on Pitching Airfoils at Supersonic Speeds. Ballistic Research Laboratories (Aberdeen Proving Ground, Md.) Report 859.
6. Wyly, Alexander, A Second Order Solution for an Oscillating Two Dimensional Supersonic Airfoil, Ph. D. Thesis, Cal Tech, 1950; Published in abridged form, Jour. Aero. Sci. Vol 19, No. 10, Oct. 1952, pp. 685 - 696, 704.
7. Van Dyke, M. D.: On Second Order Supersonic Flow Past a Slowly Oscillating Airfoil. Readers' Forum, Jour. of Aero. Sci., Jan 1953, p 61.
8. Hilton, W. F.: High-Speed Aerodynamics. Longmans, Green and Co. (New York) 1951, pp 237-240.
9. Martin, John C.: Retarded Potentials of Supersonic Flow. Quarterly of Applied Math. Vol VIII, No. 4, Jan 1951, pp 358 - 364.
10. Robinson, A.: On Source and Vortex Distributions in the Linearized Theory of Steady Supersonic Flow. Rept. No. 9, College of Aero., Cranfield (British), Oct 1947. Also published in Quart. Jour of Mech. and Appl. Math., Vol 1, Part 4, Dec. 1948, pp 408-432.
11. Heaslet, Max A., and Lomax, Harvard: The use of Source-Sink and Doublet Distributions Extended to the Solution of Boundary - Value Problems in Supersonic Flow, NACA Rep. 900, 1948.

# APPENDIX A

In ref. (9) several expressions are given for the scalar potential function of the three dimensional time dependent linearized partial differential equation of supersonic flow. In this appendix the results of ref. (9) are extended to include eq. (11). The notation of ref. (9) will be used in this appendix.

Eq. (17) of ref. (9) can be written as

$$\oint \left[ \frac{1}{R} \nabla \phi - \left( \phi - \frac{R}{\beta^2 c} \frac{\partial \phi_1}{\partial t_1} + \frac{R}{\beta^2 c} \frac{\partial \phi_2}{\partial t_2} \right) \nabla \frac{1}{R} + i \frac{M}{\beta^2 c R} \left( \frac{\partial \phi_1}{\partial t_1} + \frac{\partial \phi_2}{\partial t_2} \right) \right] \cdot \bar{n}_h da = \quad (A-1)$$

$$\iiint \left( \nabla_h^2 \phi - \frac{2V}{c^2} \frac{\partial \phi_1}{\partial \xi \partial t_1} - \frac{1}{c^2} \frac{\partial^2 \phi_1}{\partial t_1^2} - \frac{2V}{c^2} \frac{\partial \phi_2}{\partial \xi \partial t_2} - \frac{1}{c^2} \frac{\partial^2 \phi_2}{\partial t_2^2} \right) \frac{1}{R} dv$$

where the closed surface of the integral of the left side incloses the volume of the integral on the right side.

The potential function,  $\phi$ , is required to satisfy eq. (11). In this case eq. (A-1) reduces to

$$\oint \left[ \frac{1}{R} \nabla \phi - \left( \frac{R}{\beta^2 c} \frac{\partial \phi_1}{\partial t_1} + \frac{R}{\beta^2 c} \frac{\partial \phi_2}{\partial t_2} \right) \nabla \frac{1}{R} + i \frac{M}{\beta^2 c R} \left( \frac{\partial \phi_1}{\partial t_1} + \frac{\partial \phi_2}{\partial t_2} \right) \right] \cdot \bar{n}_h da = \quad (A-2)$$

$$\frac{1}{2} \iiint [(F_1 + F_2)/R] dv$$

Eq. (A-2) will be applied to a volume (denoted by  $v_0$ ) inclosed in the forward Mach cone from the point  $(x, y, z)$  (see fig. (11)). This volume is bounded by the forward Mach cone, an arbitrary surface,  $S_1$  inclosed in the forward Mach cone, and a surface given by  $\xi = x - \delta$ , where  $\delta$  is small.

The finite part of the surface integral is zero since on the Mach cone

$$t_1 = t_2$$

The surface integral over the area

$$\xi = x - \delta$$

reduces to a time independent problem in the limit as  $\delta$  approaches zero. Thus this surface integral in the limit becomes

$$\lim_{\delta \rightarrow 0} \iint \left[ \frac{1}{R} \frac{\partial \phi}{\partial n_h} - \phi \frac{\partial}{\partial n_h} \left( \frac{1}{R} \right) \right] da = -2\pi \phi(x, y, z, t)$$

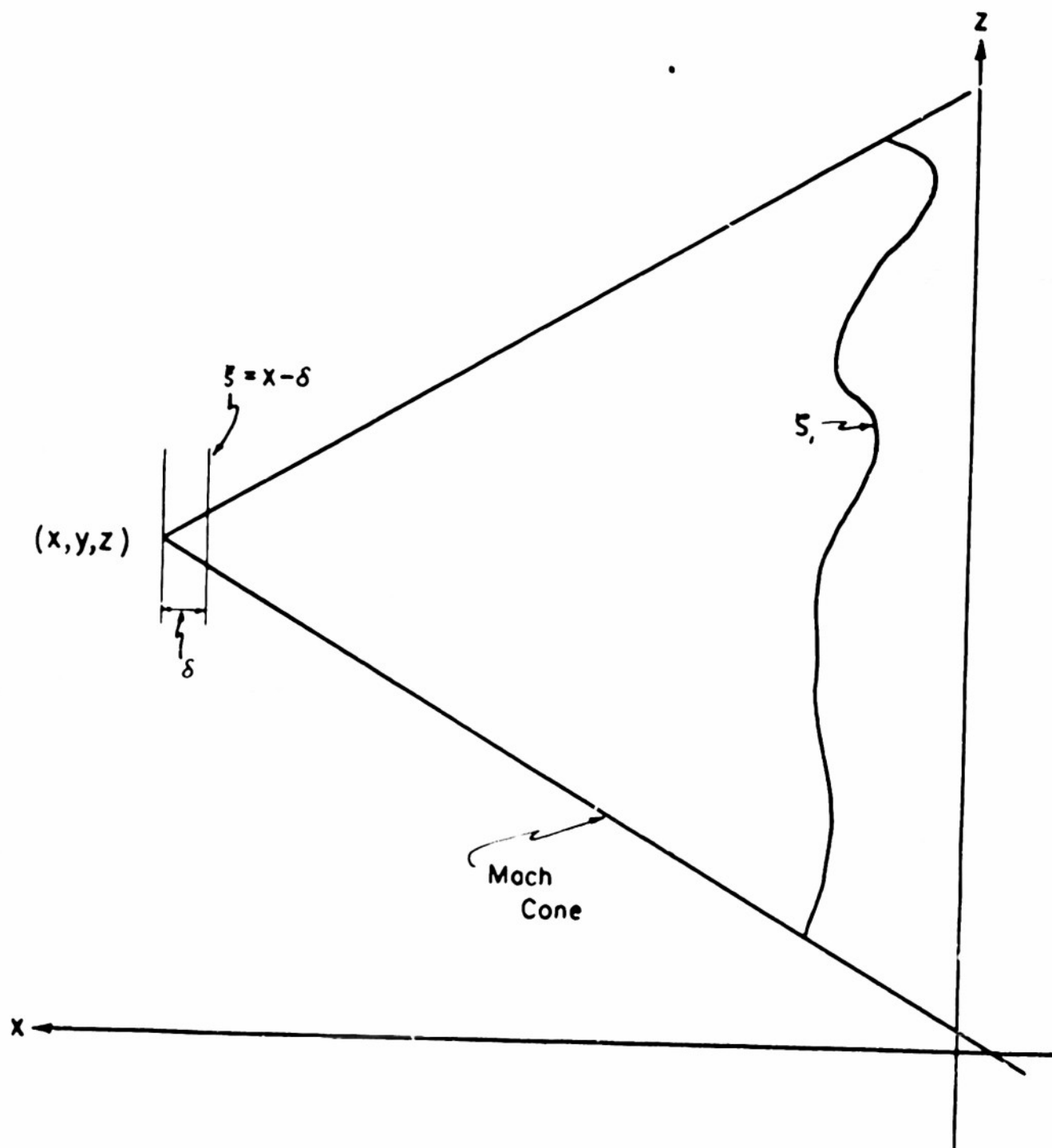


Fig. 11

The details of the above integration may be found in ref. (10) or ref. (11).

It follows from the preceding paragraph and eq. (A-2) that the potential function at the point  $(x, y, z)$  can be expressed as

$$\phi(x, y, z, t) = -\frac{1}{4\pi} \iiint_{V_0} [(F_1 + F_2)/R] dv + \frac{1}{2\pi} \iint_{S_1} \left[ \frac{1}{R} \nabla \phi - \left( \phi - \frac{R}{\beta^2 c} \frac{\partial \phi_1}{\partial t_1} + \frac{R}{\beta^2 c} \frac{\partial \phi_2}{\partial t_2} \right) \nabla \frac{1}{R} + i \frac{M}{\beta^2 c R} \left( \frac{\partial \phi_1}{\partial t_1} + \frac{\partial \phi_2}{\partial t_2} \right) \right] \vec{n} d\Omega \quad (A-3)$$

For planar problems in which the point  $(x, y, z)$  lies above the  $z = 0$  plane and the disturbing surface lies in the  $z = 0$  plane eq. (A-3) can be reduced to

$$\phi(x, y, z, t) = -\frac{1}{4\pi} \iiint_{V_0} [(F_1 + F_2)/R] dv - \frac{1}{2\pi} \iint_{S_1} \left[ (\phi_\zeta - \phi'_\zeta) \frac{1}{R} + \left( \phi - \phi' - \frac{R}{\beta^2 c} \frac{\partial \phi_1}{\partial t_1} + \frac{R}{\beta^2 c} \frac{\partial \phi'_1}{\partial t_1} + \frac{R}{\beta^2 c} \frac{\partial \phi_2}{\partial t_2} - \frac{R}{\beta^2 c} \frac{\partial \phi'_2}{\partial t_2} \right) \right] d\xi d\eta \quad (A-4)$$

where the surface  $S_1$  is the  $z = 0$  plane.

The potential function  $\phi'$  is as yet undefined. The potential function  $\phi'$  will be defined so that

$$\phi'(x, y, -z_0, t) = \phi(x, y, z_0, t)$$

In this case eq. (A-4) reduces to

$$\phi(x, y, z, t) = -\frac{1}{4\pi} \iiint_{V_0} [(F_1 + F_2)/R] dv - \frac{1}{2\pi} \iint_{S_1} \left( \frac{\partial \phi_1}{\partial t_1} + \frac{\partial \phi_2}{\partial t_2} \right) \frac{1}{R} d\xi d\eta \quad (A-5)$$



## APPENDIX B

### The Relation Between the $C_{m_a}$ Plus the $C_{m_q}$ and the Damping of Oscillating Airfoils

The damping of oscillating airfoils calculated on the basis of the linearized theory can easily be shown to be directly proportional to the sum  $C_{m_a} + C_{m_q}$  calculated by the linearized theory. The question

arises -- does this hold true for second order calculations? This question will be answered by investigating the partial differential equations and the boundary conditions for the three types of motions involved.

Eq. (1) can be used in the analysis of oscillating airfoils; however, for comparison purposes it is more convenient to express the equation in terms of axes fixed to the airfoil. The relations between the two sets of axes are (see fig. 12)

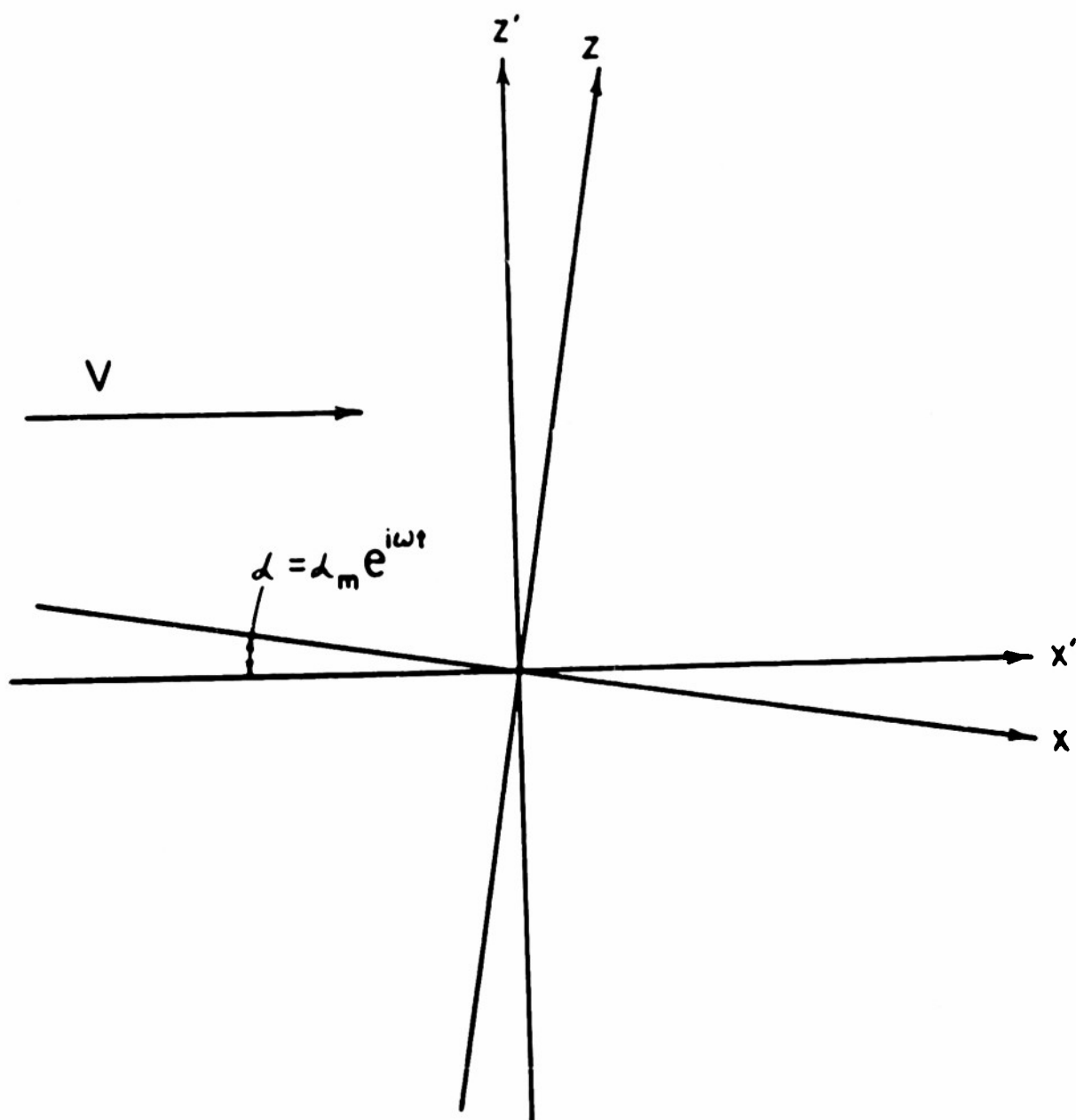
$$\begin{aligned}x &= x' \cos \alpha_m e^{i\omega t} - z' \sin \alpha_m e^{i\omega t} \\z &= x' \sin \alpha_m e^{i\omega t} + z' \cos \alpha_m e^{i\omega t} \\t &= t'\end{aligned} \tag{B-1}$$

For the axes attached to the airfoil eq. (1) becomes (to the second order)

$$\begin{aligned}-\beta^2 \Phi_{xx} + \Phi_{zz} - (2M^2/V) \Phi_{xt} - (1/c^2) \Phi_{tt} &= \alpha_m e^{i\omega t} \left[ 2M^2 \left\{ \Phi_{xz} + (\Phi_{zt}/V) - \right. \right. \\&\quad \left. \left. (i\omega z \Phi_{xx}/V) + (i\omega x \Phi_{xz}/V) + (i\omega \Phi_z/V) \right\} - \right. \\&\quad \left. (2i\omega/c^2)(z \Phi_{xt} - x \Phi_{zt}) + (\omega^2/c^2)(z \Phi_x - x \Phi_z) \right] + \\&\quad M^2 \left[ (\gamma-1)(\Phi_x + \Phi_t/V)(\Phi_{xx} + \Phi_{zz}) + 2\Phi_x \Phi_{xx} + 2\Phi_z \Phi_{xz} + \right. \\&\quad \left. (2\Phi_x \Phi_{xt}/V) + (2\Phi_z \Phi_{zt}/V) \right]\end{aligned} \tag{B-2}$$

For slowly oscillating airfoils the time variation can be expanded in the form

$$e^{i\omega t} = 1 + i\omega t - \omega^2 t^2/2 + \dots$$



The result of substituting this relation into eq. (B-2) and retaining only the terms up to first power of  $\omega$  yields:

$$\begin{aligned}
 & -\beta^2 \Phi_{xx} + \Phi_{zz} - (2M^2 \Phi_{xt}/V) - (\Phi_{tt}/c^2) = a_m 2M^2 (\Phi_{xz} + \Phi_{zt}/V) + \\
 & a_m i\omega 2M^2 \left[ t \Phi_{xz} + (t \Phi_{zt}/V) - (z \Phi_{xx}/V) + (x \Phi_{xz}/V) + \right. \\
 & \left. (\Phi_z/V) - (z \Phi_{xt}/V^2) + (x \Phi_{zt}/V^2) \right] + \\
 & M^2 \left\{ (\gamma-1) (\Phi_x + \Phi_t/V) (\Phi_{xx} + \Phi_{zz}) + 2\Phi_x \Phi_{xx} + 2\Phi_z \Phi_{xz} + \right. \\
 & \left. (2\Phi_x \Phi_{xt}/V) + (2\Phi_z \Phi_{zt}/V) \right\}
 \end{aligned} \quad (B-3)$$

The boundary condition on the airfoil surface can be expressed as

$$v \cdot \nabla s = 0$$

In this case

$$v = i(1 + \phi_x) (-\epsilon f_x) + j \left( a_m \frac{x i \omega}{V} + \phi_z + \Phi_z + a_m - a_m i\omega t \right)$$

Therefore, on the airfoil's upper surface

$$\phi_z \Big|_{z=0} = -a_m - \left[ a_m i\omega(x + Vt)/V \right] + \epsilon f_x \quad (B-4)$$

and

$$\psi_z \Big|_{z=0} = \phi_x \Big|_{z=0} + \epsilon f_x - \epsilon f_{\phi_{zz}} \Big|_{z=0} \quad (B-5)$$

The discontinuity in  $\psi$  across the leading edge Mach sheet need not be evaluated since the differential equation and the boundary conditions on the airfoil surface are sufficient to determine its value.

The first order boundary condition on the airfoil's upper surface for a constant vertical acceleration and a steady pitching is (from eq. (8), and eq. (8) of ref. (5))

$$\phi_z \Big|_{z=0} = -a - \left[ at + (qx/V) \right] + \epsilon f_x \quad (B-6)$$

Comparing eq. (B-4) and (B-6) it can be seen that if we let

$$\dot{a} = q = a_m i\omega, \quad (B-7)$$

to the first order the oscillating motion can be considered as the sum of a constant vertical acceleration and a steady pitching motion.

The second order boundary condition on the airfoil's upper surface for a constant vertical acceleration and a steady pitching is (from eq. (9), and eq. (9) of ref. (5))

$$\psi_z|_{z=0} = \epsilon \left( \phi_x^q|_{z=0} + \dot{\phi}_x^a|_{z=0} \right) = \epsilon f \left( \phi_{zz}^q|_{z=0} + \dot{\phi}_{zz}^a|_{z=0} \right) \quad (B-8)$$

where  $\phi^q$  denotes the first order solution associated with pitching, and  $\phi^a$  denotes the first order solution associated with a constant vertical acceleration. Comparing eqs. (B-5) and (B-8) it can be seen that the second order boundary condition on the airfoil surface for a slowly oscillating airfoil can be considered as the sum of a constant vertical acceleration and a steady pitching motion.

The second order partial differential equation for the combined motion of accelerating and pitching can be written as

$$\begin{aligned} -\beta^2 \Phi_{xx} + \Phi_{zz} - (2M^2 \Phi_{xt}/V) - (\Phi_{tt}/c^2) = \frac{2M^2 \alpha_m i\omega}{V} \left( x\Phi_{xz} + \Phi_z - z\Phi_{xx} + \right. \\ \left. Vt \Phi_{xz} + t\Phi_{zt} \right) + 2M^2 \left\{ \left[ (\gamma-1)/2 \right] (\Phi_x + \Phi_t/V) (\Phi_{xx} + \Phi_{zz}) + \right. \\ \left. \Phi_x \Phi_{xx} + \Phi_z \Phi_{xz} + (\Phi_x \Phi_{xt}/V) + (\Phi_z \Phi_{zt}/V) \right\} \end{aligned} \quad (B-9)$$

where  $q$  and  $\dot{a}$  have been replaced by  $\alpha_m i\omega$ . Eqs. (B-3) and (B-9) are not the same; they agree except for the terms  $\alpha_m 2M^2 \Phi_{xz}$ ,

$\alpha_m 2M^2 \Phi_{zt}/V$ ,  $\alpha_m i\omega 2M^2 z\Phi_{xt}/V^2$ , and  $\alpha_m i\omega 2M^2 x\Phi_{zt}/V^2$ .

These terms do not contribute to the part of the second order solution which is of the form  $\alpha_m i\omega \epsilon h(x,y,z,t)$ . This can be established by analyzing the form of the second order solution in the same manner that the form of the accelerating solution was analyzed previously in this paper. An analysis of the form of the second order solution for the slowly oscillating airfoil will also show that only that part of the second order solution which is of the form  $\alpha_m i\omega \epsilon h(x,y,z,t)$  will

contribute to the lifting pressure which is out of phase with the instantaneous angle of attack, and hence to the aerodynamic damping of the motion. It follows that to the second order the damping of slowly oscillating airfoils is directly proportional to  $C_{m\dot{a}} + C_{mq}$ .

# DISTRIBUTION LIST

<u>No. of Copies</u>	<u>Organization</u>
4	Chief of Ordnance Department of the Army Washington 25, D. C. Attn: ORDTB - Bal Sec
10	British - ORDTB for distribution
4	Canadian Joint Staff - ORDTB for distribution
4	Chief, Bureau of Ordnance Department of the Navy Washington 25, D. C. Attn: Re3
1	Commander Naval Proving Ground Dahlgren, Virginia
144	GM/ML 200/2 List No. 2 - Parts A. C. DA
5	Director Armed Services Technical Information Agency Documents Service Center Knott Building Dayton 2, Ohio Attn: DSC-SA
1	Dr. Allen E. Puckett Hughes Aircraft Company Florence Avenue at Teal Street Culver City, California
1	National Advisory Committee for Aeronautics Ames Laboratory Moffett Field, California Attn: Dr. Milton D. Van Dyke
4	National Advisory Committee for Aeronautics Langley Field, Virginia Attn: Dr. Adolf Busemann (1 cy) Mr. Frank S. Malvestuto (1 cy) Mr. Leonard Sternfield (1 cy) Mr. C. E. Brown (1 cy)